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The Preferred Null Symmetries of a Null Electromagnetic Field Interacting with a Null Free Gravitational Field

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The real null vector $\mathbf{l}^a$ of the Newman–Penrose formalism is preferred to correspond to a geometrical symmetry as well as a dynamical symmetry. The 16 types of geometrical symmetries expressed through the vanishing of the Lie derivatives of certain tensor fields with respect to $\mathbf{l}^a$ are examined separately. Two types of dynamical symmetries are imposed simultaneously on $\mathbf{l}^a$: A null electromagnetic field and a null gravitational field are both chosen to have the same propagation vector $\mathbf{l}^a$. By adopting freedom conditions on $\mathbf{l}^a$, it is shown that the symmetries of the null electromagnetic field are shared neither by the free gravitational field nor by the gravitational potentials. In fact the following five preferred null symmmetries are found to be proper: motion, affine collineation, special curvature collineation, curvature collineation, and Ricci collineation. The scalars characterizing the coupled fields are found to be constant with respect to $\mathbf{l}^a$.

1. INTRODUCTION

In the general theory of relativity the curvature tensor describing the gravitational field consists of two parts, *viz.*, the matter part and the free gravitational part (Szekeres, 1964). The interaction between the two parts is described through the Bianchi identities. Until the advent of the Newman–Penrose formalism in 1962, the full implication of the 24 Bianchi identities could not be exploited. In fact the twice contracted Bianchi identities, which are just four in number, prompted Einstein to identify the geometric entity (divergence-free tensor), $R^{ab}-\frac{1}{2}Rg^{ab}$ with the dynamic

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entity T^{ab}, the stress energy momentum tensor of ponderable matter. For a given distribution of matter, the construction of gravitational potentials satisfying Einstein's field laws is the principal aim of all investigations in gravitation physics. Traditionally this has been accomplished by imposing symmetries on the geometry compatible with the dynamics of the chosen distribution of matter. Quite often the geometrical symmetries are expressible through the vanishing of the Lie derivative (of certain tensors) with respect to a vector. This vector may be timelike, spacelike, or null. In this paper, we confine ourselves to the one-parameter group of symmetries comprising of motions as well as collineations with respect to a special null vector. In a series of papers, Davis and his collaborators (Davis, 1974a, 1974b, 1977; Davis and Moss, 1963, 1970; Davis and York, 1969; (Davis et al., 1976) have identified 16 symmetries for the gravitational field and obtained the corresponding weak conservation laws as the integrals of the geodesic equations.

In the absence of matter, the free null gravitational field with twisting rays admits at most one symmetry (Collinson, 1969), while for the non-null gravitational field the curvature collineation (CC) degenerates to conformal motion (Collinson, 1970). Collinson and Dodd (1971) have determined the Killing symmetries of the stationary axisymmetric empty space-times. The homothetic motions for algebraically special free gravitational fields in vacuum have been studied by Halford and Kerr (1980). Recently, McIntosh (1980) has observed that there are very few space-times for which the CC are proper (i.e., do not degenerate to conformal motions).

In the absence of the free gravitational field, a non-Einstein space with a nonzero scalar curvature does not admit a proper CC because it degenerates to motion (Katzin, 1970). In these conformally flat spaces the CC and the special conformal motion are equivalent (Levine and Katzin, 1970).

Different types of matter distributions compatible with the geometrical symmetries have attracted the attention of several investigators. Oliver and Davis (1977) have studied the timelike symmetries, with special reference to conformal motions and family of contracted Ricci collineations, for the space-times filled with perfect fluid. The nonexistence of proper homothetic spacelike motions in a space of constant curvature permeated by a magnetofluid has been established by Prasad and Sinha (1979). Eardley (1964) has considered the perfect fluids which are homogeneous in cosmology and investigated that only a tilted cosmology can have a nontrivial homothetic vector. The perfect fluid space-times including electromagnetic fields admitting symmetry mappings belonging to the family of contracted Ricci collineations are studied by Norris et al. (1977).

Khade and Radhakrishna (1974) have introduced the concept of Einstein collineation (which is different from the 16 symmetries) and investigated the preferred symmetries along the propagation vector and the polarization vector. Similar symmetries have been studied for the magnetofluid by Asgekar and Date (1975–76) and for a fluid collapsing with neutrino emission by Radhakrishna and Rao (1975–76). Lukacs et al. (1980) have recently proved the theorem that "when an electro-vacuum field admits a null Killing vector $\mathbf{l}^a$, then either the principal null directions of the Maxwell field are Lie-propagated along $\mathbf{l}^a$ or else the Maxwell field is degenerate or third, $\mathbf{l}^a$ is hypersurface orthogonal." Tariq and Tupper (1977) have confined themselves to one symmetry (CC) for the electromagnetic field and showed that it degenerates to a conformal motion except for the null electromagnetic field and for the null gravitational field.

The authors are not aware of any attempt at investigating all the 16 symmetries for a given distribution of matter. Accordingly the simplest type of a nonempty field interacting with the simplest type of a free gravitational field is considered and all the symmetries of such a coupling are examined with respect to the real null vector $\mathbf{l}^a$ of the Newman–Penrose (NP) tetrad.

When a vector of a tetrad characterizes a geometrical symmetry or a dynamical symmetry or both, it is termed a preferred vector. For example (Synge, 1972), when a null vector is identified as the history of a massless particle with given 4-momentum tangent to a null line, then it is called a preferred null vector. Yet another instance of a preferred vector is when it is chosen as a Killing vector (Geroch et al., 1973). In this paper, the vector $\mathbf{l}^a$ of the NP tetrad is given a preferential role. It is not only the propagation vector of the null electromagnetic field but also the propagation vector of the null gravitational field.

The procedure adopted in this paper for getting the necessary and sufficient conditions for the existence of the different symmetries can be stated as follows: The tensorial differential equations characterizing a symmetry are transcribed into the NP formalism as algebraic equations in the form of linear combinations of the outer products of the tetrad Z_a^α with coefficients involving spin coefficients and their intrinsic derivatives. The required necessary and sufficient conditions follow (with effortless ease!) just by equating to zero each one of the coefficients. For instance, if Γ^a_{bc} is the Christoffel symbol and

$$\underset{\mathbf{l}}{\mathscr{L}}\Gamma^a_{bc} = A^\alpha{}_{\beta\gamma} Z^a_{(\alpha)} Z_b^{(\beta)} Z_c^{(\gamma)}$$

then

$$\underset{\mathbf{l}}{\mathscr{L}}\Gamma^a_{bc} = 0 \Leftrightarrow A^\alpha{}_{\beta\gamma} = 0$$

which is a sequel to the fact that the vectors $Z^a_{(\alpha)}(\alpha = 1,2,3,4)$ form a basis and the outer products $Z^a_{(\alpha)} Z_b^{(\beta)} Z_c^{(\gamma)}$ are independent. Here $A^\alpha{}_{\beta\gamma}$ are functions of spin coefficients and their intrinsic derivatives.

After the investigation of these preferred null symmetries, it is shown that the symmetries of the null electromagnetic field are different from either the free gravitational field or the gravitational potentials. In fact the following five preferred null symmetries are found to be proper: motion, affine collineation, special curvature collineation, curvature collineation and Ricci collineation.

In Section 2, the null electromagnetic field and the null gravitational field (i.e., the coupled fields) which are considered in this paper are delineated, while in Section 3, the conditions satisfied for the interaction of these two fields are listed. In the next section the five proper null symmetries are investigated. A discussion of the scope of the results is given in the last section.

2. THE COUPLED FIELDS

Let $Z_a^\alpha = (\mathbf{l}_a, n_a, m_a, \bar{m}_a)$ be an orthonormal null tetrad. Here $\mathbf{l}_a, n_a$ are real null vectors and m_a is a complex null vector with $\bar{m}_a$ as its conjugate. These null vectors satisfy the following conditions:

$$\mathbf{l}_a n^a = -m_a \bar{m}^a = 1$$

$$\mathbf{l}_a m^a = \mathbf{l}_a \bar{m}^a = n_a m^a = n_a \bar{m}^a = 0$$

$$\mathbf{l}_a \mathbf{l}^a = n_a n^a = m_a m^a = \bar{m}_a \bar{m}^a = 0$$

The null electromagnetic field with $\mathbf{l}^a$ as the eigenvector of the stress tensor T^{ab} is characterized by (Debney and Zund, 1971)

$$T_{ab} = \tfrac{1}{2}|\phi|^2 \mathbf{l}_a \mathbf{l}_b \tag{1}$$

where $\phi = 2F_{ab}\bar{m}^a n^b$ and

$$F_{ab} = \phi \mathbf{l}_{[a} m_{b]} + \bar{\phi} \mathbf{l}_{[a} \bar{m}_{b]}$$

is the electromagnetic field tensor with $\mathbf{l}_{[a} m_{b]} = \tfrac{1}{2}(\mathbf{l}_a m_b - \mathbf{l}_b m_a)$. Since $F_{ab}\mathbf{l}^b = 0$, we identify $\mathbf{l}^a$ as the propagation vector of the field. The corresponding Maxwell's equations are (Carmeli, 1977)

$$D\phi = (\rho - 2\varepsilon)\phi$$

$$\delta\phi = (\tau - 2\beta)\phi$$

$$\kappa = \sigma = 0 \tag{2}$$

where κ, σ, ρ, ε, τ, and β are the Newman–Penrose spin coefficients.

The null gravitational field with $\mathbf{l}^a$ as the propagation vector is characterized by (Szekeres, 1964)

$$C_{abcd} = -4\,\mathrm{Re}\{\psi \mathbf{l}_{[a} m_{b]} \mathbf{l}_{[c} m_{d]}\} \tag{3}$$

where C_{abcd} is the Weyl conformal curvature tensor. Here the Weyl scalar ψ can be identified as

$$\psi = -C_{pqrs}\bar{m}^p n^q \bar{m}^r n^s$$

We note that $C_{abcd}\mathbf{l}^a = 0$, $C_{abcd}\mathbf{l}^d = 0$. Consequently $\mathbf{l}^a$ can be identified as the propagation vector for the null gravitational field. The Riemann Christoffel curvature tensor and the Weyl projective curvature tensor for the coupled field are, respectively,

$$R^a_{bcd} = \tfrac{1}{2}(\delta^a_d R_{bc} - \delta^a_c R_{bd} + g_{bc}R^{a\cdot}{}_d - g_{bd}R^{a\cdot}{}_c) - R/6(\delta^a_c g_{bd} - \delta^a_d g_{bc}) - \mathrm{Re}\{\psi V^{a\cdot}{}_b V_{cd}\} \tag{4}$$

$$W^a_{bcd} = R^a_{bcd} + \tfrac{1}{3}(\delta^a_c R_{bd} - \delta^a_d R_{bc}) \tag{5}$$

where

$$R_{ab} = \chi(T_{ab} - \tfrac{1}{2}Tg_{ab})$$

χ is constant, and

$$V_{ab} = 2\mathbf{l}_{[a}m_{b]}$$

3. THE FREEDOM CONDITIONS AND THE IDENTITIES

The investigation of the null symmetries in the later sections is carried out under certain conditions which are enumerated below:

(a) *Freedom Conditions with Respect to* $\mathbf{l}^a$ (Chandrasekhar, 1979). (1) If the tetrad Z^a_a is parallelly propagated with respect to $\mathbf{l}^a$, then

$$\kappa = \pi = \varepsilon = 0 \tag{6a}$$

(2) If the congruence $\mathbf{l}^a$ is equal to a gradient field, then

$$\rho = \bar{\rho}, \qquad \tau = \bar{\alpha} + \beta \tag{6b}$$

(b) *Bianchi Identities for the Coupled Fields*. The interaction between the null electromagnetic field and the null gravitational field is expressed through the Bianchi identities. By virtue of the freedom conditions and (1)–(3), the 11 Bianchi identities reduce to

$$\bar{\delta}(\phi\bar{\phi}) - \delta\psi = (4\beta - \tau)\psi - \bar{\gamma}\phi\bar{\phi} \tag{7a}$$

$$\rho = 0 \tag{7b}$$

$$D\phi = 0 \tag{7c}$$

$$D\psi = 0 \tag{7d}$$

where $D = \mathbf{l}^a(\partial/\partial x^a)$, $\Delta = n^a(\partial/\partial x^a)$, and $\delta = m^a(\partial/\partial x^a)$. Thus from (7c), (7d) the Maxwell scalar ϕ and the Weyl scalar ψ are conserved with respect to $\mathbf{l}^a$. We note that (7b) simplifies (6b).

(c) *The NP Equations for the Coupled Fields*. The 18 Ricci identities known as the NP equations are enumerated in Flaherty (1976). Using $\rho = 0$, freedom conditions and the defining relations (1), (3) of the fields under question, these NP equations yield the following 15 relations:

$$D\tau = D\alpha = D\beta = D\lambda = D\mu = 0$$

$$D\gamma = \tau\alpha + \bar{\tau}\beta$$

$$D\nu = \bar{\tau}\mu + \tau\lambda$$

$$\delta\lambda - \bar{\delta}\mu = \mu\bar{\tau} + \lambda(\tau - 4\beta)$$

$$\delta\gamma - \Delta\beta = \mu\tau + \alpha\bar{\lambda} - \beta(\gamma - \bar{\gamma} - \mu)$$

$$\delta\tau = 2\beta\tau$$

$$\bar{\delta}\tau = 2\alpha\tau$$

$$\Delta\alpha - \bar{\delta}\gamma = -(\tau + \beta)\lambda + (\bar{\gamma} - \gamma - \bar{\mu})\alpha$$

$$\Delta\lambda - \bar{\delta}\nu = -(\mu + \bar{\mu})\lambda - (3\gamma - \bar{\gamma})\lambda + 2\alpha\nu - \psi$$

$$\delta\alpha - \bar{\delta}\beta = \alpha\bar{\alpha} + \beta\bar{\beta} - 2\alpha\beta$$

$$\delta\nu - \Delta\mu = \mu^2 + \lambda\bar{\lambda} + (\gamma + \bar{\gamma})\mu - 2\beta\bar{\nu} + \phi\bar{\phi}$$

(d) *The Commutation Relations for the Coupled Fields*. Adopting freedom conditions and $\rho = 0$, the four commutation relations become

$$(\Delta D - D\Delta) = (\gamma + \bar{\gamma})D - \tau\bar{\delta} - \bar{\tau}\delta$$

$$(\delta D - D\delta) = \tau D$$

$$(\delta\Delta - \Delta\delta) = -\bar{\nu}D + \bar{\lambda}\bar{\delta} + (\mu - \gamma + \bar{\gamma})\delta$$

$$(\delta\bar{\delta} - \bar{\delta}\delta) = -(\bar{\mu} - \mu)D + (\bar{\alpha} - \beta)\bar{\delta} + (\bar{\beta} - \alpha)\delta$$

4. THE NULL SYMMETRIES OF THE COUPLED FIELDS

The null symmetry of a tensor field $\Omega^{\cdots a \cdots}_{\cdots b \cdots}$ with respect to $\mathbf{l}^a$ is characterized by the functional form invariance of $\Omega^{\cdots a \cdots}_{\cdots b \cdots}$ with respect to the infinitesimal point transformation

$$x'^i = x^i + \mathbf{l}^i \delta t, \qquad \mathbf{l}^i \mathbf{l}_i = 0$$

where t is a parameter. This fact is expressed through the differential equations

$$\underset{\mathbf{l}}{\mathscr{L}}\Omega^{\cdots a \cdots}_{\cdots b \cdots} = 0$$

where $\mathscr{L}_{\mathbf{l}}$ represents the Lie derivative operator along $\mathbf{l}^a$.

I. Null Motions. Null motions are characterized by the Killing equations

$$\underset{\mathbf{l}^a}{\mathscr{L}} g_{ab} = 0 \tag{8}$$

or $\mathbf{l}_{a;b} + \mathbf{l}_{b;a} = 0$. In the Newman–Penrose formalism, we have (Chandrasekhar, 1979)

$$\begin{aligned}\mathbf{l}_{a;b} = {} & (\gamma + \bar{\gamma})\mathbf{l}_a\mathbf{l}_b - (\alpha + \bar{\beta})\mathbf{l}_a m_b - (\bar{\alpha} + \beta)\mathbf{l}_a \bar{m}_b + (\varepsilon + \bar{\varepsilon})\mathbf{l}_a n_b \\ & - \bar{\tau} m_a \mathbf{l}_b + \bar{\sigma} m_a m_b + \bar{\rho} m_a \bar{m}_b - \bar{\kappa} m_a n_b - \tau \bar{m}_a \mathbf{l}_b + \rho \bar{m}_a m_b \\ & + \sigma \bar{m}_a \bar{m}_b - \kappa \bar{m}_a n_b\end{aligned}$$

Hence the equations (8) on using (6a), (6b) become

$$(\gamma + \bar{\gamma})\mathbf{l}_a\mathbf{l}_b + 2\rho m_{(a}\bar{m}_{b)} + \left(\bar{\sigma} m_a m_b - 2\bar{\tau}\mathbf{l}_{(a}m_{b)}\right) + \text{C.C.} = 0 \tag{9}$$

where $\mathbf{l}_{(a}m_{b)} = \frac{1}{2}(\mathbf{l}_a m_b + \mathbf{l}_b m_a)$ and C.C. indicates the complex conjugate of the preceding term.

The equations (9) are valid iff

$$\gamma + \bar{\gamma} = 0, \qquad \sigma = \tau = \rho = 0 \tag{10}$$

Hence (10) are the necessary and sufficient conditions for the existence of the motion with respect to the null vector ray $\mathbf{l}^a$.

The Null Symmetries which Degenerate into the Null Motions. The null conformal motions are defined by

$$\underset{\mathbf{l}}{\mathscr{L}} g_{ab} = 2A g_{ab} \tag{11}$$

where A is a scalar function. The freedom conditions (6a), (6b) together with (11) imply that $A = 0$ and hence this symmetry degenerates into the null motion. (Note that if the freedom conditions are not used, then $A \neq 0$ and so the degeneracy can be avoided.)

It follows that the special conformal null motion ($A_{;ab} = 0$) and the null homothetic motion (A is constant) are also improper. Here a symmetry is termed proper, if it does not degenerate into any other symmetry.

II. Affine Null Collineation. This symmetry is defined by (Katzin et al., 1969)

$$\underset{\mathbf{l}}{\mathscr{L}}\Gamma^a_{bc} = 0, \text{ i.e., } \mathbf{l}^a_{;cb} + R^a_{\cdot cmb}\mathbf{l}^m = 0 \tag{12}$$

For the null gravitational fields with $\mathbf{l}^a$ as the propagation vector, we have

$$R^{a}_{\cdot cmb}\mathbf{l}^m = 0 \tag{13}$$

In NP formalism, the second (-order) covariant derivative of $\mathbf{l}^a$ after imposing the freedom conditions (6a), (6b) reduces to

$$\begin{aligned}
\mathbf{l}^a_{;cb} = {} & \mathbf{l}^a\mathbf{l}_b\mathbf{l}_c\left[\Delta(\gamma+\bar{\gamma})+2(\gamma+\bar{\gamma})^2-2\bar{\nu}\bar{\tau}-2\nu\tau\right] \\
& + (\mathbf{l}^a\mathbf{l}_b n_c + n^a\mathbf{l}_b\mathbf{l}_c)(2\tau\bar{\tau}) + \mathbf{l}^a n_b \mathbf{l}_c[D(\gamma+\bar{\gamma})] \\
& + \{\mathbf{l}^a\mathbf{l}_b m_c[-\Delta\bar{\tau}-\bar{\tau}(3\gamma+\bar{\gamma})+\bar{\sigma}\bar{\nu}+\rho\nu] \\
& \quad + \mathbf{l}^a m_b \mathbf{l}_c[-\bar{\delta}(\gamma+\bar{\gamma})-2\bar{\tau}(\gamma+\bar{\gamma}-\bar{\mu})+2\tau\lambda] \\
& \quad + \mathbf{l}^a m_b m_c[\bar{\delta}\bar{\tau}+\bar{\sigma}(\gamma+\bar{\gamma}-\bar{\mu})+2\alpha\bar{\tau}-\lambda\rho] \\
& \quad + \mathbf{l}^a m_b \bar{m}_c[\bar{\delta}\tau+\rho(\gamma+\bar{\gamma}-\bar{\mu})+2\bar{\beta}\tau-\lambda\sigma] \\
& \quad + \mathbf{l}^a m_b n_c[-\rho\bar{\tau}-\bar{\sigma}\tau]+\mathbf{l}^a n_b m_c[-D\bar{\tau}] \\
& \quad + n^a\mathbf{l}_b m_c[-\rho\bar{\tau}-\tau\bar{\sigma}]+n^a m_b\mathbf{l}_c[-\rho\bar{\tau}-\bar{\sigma}\tau]+n^a m_b m_c[2\rho\bar{\sigma}] \\
& \quad + n^a m_b\bar{m}_c[\rho^2+\sigma\bar{\sigma}]+m^a\mathbf{l}_b\mathbf{l}_c[-\Delta\bar{\tau}-\bar{\tau}(3\gamma+\bar{\gamma})+\bar{\sigma}\bar{\nu}+\rho\nu] \\
& \quad + m^a\mathbf{l}_b n_c[-\bar{\sigma}\tau-\rho\bar{\tau}]+m^a n_b\mathbf{l}_c[-D\bar{\tau}] \\
& \quad + m^a\mathbf{l}_b m_c[\Delta\bar{\sigma}+2\bar{\tau}\bar{\tau}+2\bar{\sigma}(\gamma-\bar{\gamma})] \\
& \quad + m^a m_b\mathbf{l}_c[\bar{\delta}\bar{\tau}+\bar{\sigma}(\gamma+\bar{\gamma}-\bar{\mu})+2\alpha\bar{\tau}-\rho\lambda] \\
& \quad + m^a\mathbf{l}_b\bar{m}_c[\Delta\rho+2\tau\bar{\tau}]+m^a m_b m_c[-\bar{\delta}\bar{\sigma}-4\alpha\bar{\sigma}] \\
& \quad + m^a\bar{m}_b\bar{m}_c[-\delta\rho-\rho\tau-\sigma\bar{\tau}] \\
& \quad + m^a\bar{m}_b m_c[-\delta\bar{\sigma}-2\rho\bar{\tau}+2\bar{\sigma}(\bar{\alpha}-\beta)] \\
& \quad + m^a m_b\bar{m}_c[-\bar{\delta}\bar{\rho}-\bar{\sigma}\tau-\rho\bar{\tau}]+m^a n_b\bar{m}_c[D\rho] \\
& \quad + m^a\bar{m}_b n_c[\sigma\bar{\sigma}+\rho^2]+m^a n_b m_c[D\bar{\sigma}]+m^a m_b n_c[2\bar{\sigma}\rho] \\
& \quad + \bar{m}^a m_b\mathbf{l}_c[\bar{\delta}\tau+\rho(\gamma+\bar{\gamma}-\bar{\mu})+2\bar{\beta}\tau-\sigma\lambda]\}+\text{C.C.}
\end{aligned} \tag{13'}$$

On using (1), (3) and (13′) the relations (12) reduce for the coupled fields to

$$\sigma=\rho=\tau=0 \qquad \text{and} \qquad \Delta(\gamma+\bar{\gamma})+2(\gamma+\bar{\gamma})^2=0 \tag{14}$$

This completely characterizes the affine null collineations.

Remark. This symmetry is proper since $\gamma+\bar{\gamma}\neq 0$ (in other words $\mathbf{l}_{a;b}\neq 0$) while the homothetic motion demands that $\gamma+\bar{\gamma}=0$.

Symmetries which Degenerate to the Affine Null Collineations. Special null geodesic null collineation is defined by

$$\underset{\mathbf{l}}{\mathscr{L}}\Gamma^a_{bc}=g_{bc}g^{aq}A_{;q},\qquad A_{;rs}=0 \tag{15}$$

These equations are satisfied iff $DA=\Delta A=\bar{\delta}A=\delta A=0$, besides (14). It follows that $A=\text{const}$ and so $A_{;r}=0$. This shows the degeneracy of the collineation to affine null collineation.

It can be shown that the following five symmetries also reduce to the affine null collineation:

(i) Special conformal null collineation:

$$\underset{\mathbf{l}}{\mathscr{L}}\Gamma^a_{bc}=\delta^a_bA_{;c}+\delta^a_cA_{;b}-g_{bc}g^{ak}A_{;k},\qquad A_{;bc}=0$$

(ii) Conformal null collineation:

$$\underset{\mathbf{l}}{\mathscr{L}}\Gamma^a_{bc}=\delta^a_bA_{;c}+\delta^a_cA_{;b}-g_{bc}g^{ak}A_{;k}$$

(iii) Null Geodesic null collineation:

$$\underset{\mathbf{l}}{\mathscr{L}}\Gamma^a_{bc}=g_{bc}g^{ak}A_{;k}$$

(iv) Special projective null collineation:

$$\underset{\mathbf{l}}{\mathscr{L}}\Gamma^a_{bc}=\delta^a_bA_{;c}+\delta^a_cA_{;b},\qquad A_{;bc}=0$$

(v) Projective null collineation:

$$\underset{\mathbf{l}}{\mathscr{L}}\Gamma^a_{bc}=\delta^a_bA_{;c}+\delta^a_cA_{;b}$$

III. Special Curvature Null Collineations. This symmetry is characterized by

$$\left(\underset{\mathbf{l}}{\mathscr{L}}\Gamma^a_{bc}\right)_{;d}\equiv\left(\mathbf{l}^a_{;cb}+R^a{}_{\cdot cmb}\mathbf{l}^m\right)_{;d}=0 \tag{16}$$

The relations (16) after a considerable simplification and on using (1), (3), (7b), (13′) and freedom conditions (6a), (6b), reduce to

$$\sigma=\rho=\tau=0,\qquad DF=\delta F=0,\qquad \text{and } \Delta F+3F(\gamma+\bar{\gamma})=0 \tag{17}$$

where

$$F\equiv\Delta(\gamma+\bar{\gamma})+2(\gamma+\bar{\gamma})^2$$

This completely characterizes the special curvature null collineation.

Remark. The special curvature null collineation for the coupled fields is proper, since $\Delta(\gamma+\bar{\gamma})+2(\gamma+\bar{\gamma})^2 \neq 0$. For the affine null collineation, it is necessary that $\Delta(\gamma+\bar{\gamma})+2(\gamma+\bar{\gamma})^2 = 0$.

IV. Curvature Null Collineation. This symmetry is defined by

$$\underset{\mathbf{l}}{\mathscr{L}} R^{a}_{\cdot bcd} = 0 \tag{18}$$

For the coupled fields under question the equations (18) are valid when (and only when)

$$\sigma = \rho = \tau = 0, \qquad D|\phi| = 0, \qquad \text{and } D\psi = 0 \tag{19}$$

Hence (19) are the necessary and sufficient conditions for the existence of the curvature null collineation with respect to the null ray $\mathbf{l}^a$.

Remarks. (1) The curvature null collineation is proper since there is no restriction on $\gamma+\bar{\gamma}$ as in the special curvature null collineation.

(2) The following two symmetries are improper, since they degenerate into curvature null collineation: (i) Weyl projective null collineation,

$$\underset{\mathbf{l}}{\mathscr{L}} W^{a}_{\cdot bcd} = 0$$

(ii) The Weyl conformal null collineation,

$$\underset{\mathbf{l}}{\mathscr{L}} C^{a}_{\cdot bcd} = 0$$

V. Ricci Null Collineation. It is characterized by

$$\underset{\mathbf{l}}{\mathscr{L}} R_{ab} = 0 \tag{20}$$

This symmetry is identically satisfied for the coupled fields on using freedom conditions (6a), (6b) since

$$\underset{\mathbf{l}}{\mathscr{L}} R_{ab} = \mathbf{l}_a \mathbf{l}_b [DP + 2P(\varepsilon+\bar{\varepsilon})] - P\bar{\kappa}\mathbf{l}_a m_b - P\kappa \mathbf{l}_a \bar{m}_b$$
$$- P\bar{\kappa} m_a \mathbf{l}_b - P\kappa \bar{m}_a \mathbf{l}_b$$

where $P = \frac{1}{2}|\phi|^2$. We have $\kappa = \varepsilon = 0$, due to (6a). $DP = 0$, due to $D\phi = 0$ from (7c).

Remarks. (1) This symmetry is proper since $\tau \neq 0$. The curvature null collineation demands that $\tau = 0$.

(2) The family of contracted Ricci collineation with respect to the null congruence $\mathbf{l}^a$ (which is defined by $g^{ab}\mathscr{L}_\mathbf{l} R_{ab} = 0$) is also identically satisfied for the coupled fields. So this symmetry is also improper as it degenerates into the Ricci null collineation.

(3) The symmetry of the null electromagnetic field defined by $\mathscr{L}_\mathbf{l} T_{ab} = 0$ is identical to the Ricci null collineation due to the fact that the stress-energy tensor for the electromagnetic field is tracefree.

(4) The symmetries of the null electromagnetic field are different from the symmetries of either the free gravitational field or the gravitational potentials, since motion, curvature collineation, and Ricci collineation are proper.

5. DISCUSSION

The investigation of the null symmetries when the null electromagnetic field is chosen as

$$T_{ab} = \tfrac{1}{2}|\phi|^2 n_a n_b$$

where $\phi = 2F_{ab}l^a m^b$,

$$F_{ab} = -\phi_0 n_{[a}\overline{m}_{b]} - \overline{\phi}_0 n_{[a} m_{b]}$$

and the null electromagnetic field (with n^a as the propagation vector) is characterized by

$$C_{abcd} = -4\,\mathrm{Re}\{\psi n_{[a}\overline{m}_{b]} n_{[c}\overline{m}_{d]}\}$$

$$\psi = -C_{abcd} l^a m^b l^c m^d$$

can be carried out analogous to the procedures adopted here and the results are similar.

Owing to the absence of convenient expressions for the Weyl tensor in the case of Petrov type I, II, D, and III, the investigation of the symmetries becomes complicated when the interaction with the electromagnetic fields (both a non-null and null) is examined.

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